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Minimal submanifolds with a parallel or a harmonic p-form

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Abstract

The purpose of this paper is to study the relations between the existence of minimal immersions of a Riemannian manifold M into another and some structural or topological properties of M. The properties on M which we consider are the existence of a parallel or a harmonic p-form.

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1. Introduction

The purpose of this paper is to obtain some nonexistence results about minimal submanifolds. Let (M^m, g) be an m-dimensional Riemannian manifold isometrically immersed by ϕ in an n-dimensional Riemannian manifold (N^n, h) $(n > m)$. The Gauss equation allows us to obtain rigidity results in terms of geometry of (M^m, g) and (N^n, h). For example, as a first consequence of the Gauss equation, we get the following well known inequality in each point x of (M^m, g):

$$|H(x)|^2 \geq \left(\frac{1}{m}\right)\left(\frac{\mathrm{Scal}(x)}{m} - (m-1)\bar{K}^1(\phi(x))\right), \qquad (1)$$

where $|H(x)|^2$ and $\mathrm{Scal}(x)$ are, respectively, the square of the mean curvature of ϕ and the scalar curvature of (M^m, g) at x and $\bar{K}^1(\phi(x))$ is the largest sectional curvature of (N^n, h)

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at $\phi(x)$. In particular, if $\bar{K}^1$ has an upper bound and if Scal $> m(m-1)\bar{K}^1_{\max}$ (where $\bar{K}^1_{\max} = \max_N(\bar{K}^1)$) for at least a point of (M^m, g), there is no minimal immersion of (M^m, g) into (N^n, h).

Many other results were obtained, by assuming that (M^m, g) is endowed with some particular structures or topological properties (see, for instance, [1,3,11]). First recall the results of Sampson [11] and Dajczer and Rodriguez [3]. They proved that there is no minimal immersion of an m-dimensional Kaehlerian manifold ($m \geq 4$) into a Riemannian manifold of negative constant sectional curvature. Later, El Soufi [5] obtained a generalization of this result by assuming a pinching of the sectional curvature of (N^n, h) and Hernandez [8] obtained the same conclusion under the negativity of the complex sectional curvature of (N^n, h). More recently, El Soufi and Petit [6] extend this result in the case where (M^m, g) is not necessarily Kaehlerian but has a parallel 2-form and where the isotropic curvature of (N^n, h) is negative (recall that the isotropic curvature of a Riemannian manifold is the restriction of the complex sectional curvature to isotropic tangent planes [10]).

Section 2 of the present paper deals with some preliminaries. In Section 3, we consider the general case where (M^m, g) has a parallel p-form and we prove (Theorem 3.1) that if (N^n, h) satisfies a curvature pinching condition, then there is no minimal immersion from (M^m, g) into (N^n, h). This is the generalization of the result of El Soufi stated in [5] for the case where (M^m, g) is Kaehlerian. Note that this theorem as well as the other results recalled above are of interest only if the sectional curvature of (N^n, h) is negative. However, in Theorem 3.2, we obtain the same conclusion with a new pinching condition for the case where (N^n, h) is not necessarily of negative sectional curvature but has a negative smallest sectional curvature. In Theorem 3.3, we study the particular case where (N^n, h) is the complex hyperbolic space $\mathbb{CH}^n(c)$ with constant holomorphic curvature equal to c and we prove that there is no totally real minimal immersion of a Riemannian manifold (M^m, g) with a parallel p-form into $\mathbb{CH}^n(c)$.

The compact manifolds with a parallel p-form are a particular case of manifolds with a harmonic p-form (or a nonzero pth Betti number $b_p(M)$). In Section 4, we prove (Theorems 4.1 and 4.2) that for any compact manifold (M^m, g) with $b_p(M) \neq 0$ and isometrically immersed in a Riemannian manifold (N^n, h), there exists at least a point x of M so that

$$\frac{m}{\sqrt{p}}\left(\frac{p-1}{p}\right)|B(x)||H(x)| \geq k(x) - \left(\frac{p-1}{p}\right)((m-1)\bar{K}^1 + \bar{\rho}^1)(\phi(x))$$

and

$$m\left(\frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}}\right)|B(x)||H(x)| \geq \mathrm{Scal}(x) - (m-2)((m-1)\bar{K}^1 + \bar{\rho}^1)(\phi(x)),$$

where $|B(x)|$, $k(x)$ and $\bar{\rho}^1(\phi(x))$ denote, respectively, the norm of the second fundamental form of ϕ, the smallest eigenvalue of the Ricci curvature of (M^m, g) at x and the largest eigenvalue of the curvature operator of (N^n, h) at $\phi(x)$. El Soufi proved the first inequality for $p = 2$ in [5] and the second for $p = 2$ but only for $m = 4$. The first is optimal for the usual standard minimal embeddings of the Clifford torus and of the complex projective space in the sphere. These inequalities will be a consequence of a new lower bound of the curvature term in the Weitzenböck formula for p-forms (see relation (5) and Propositions 4.1 and 4.2).

As a consequence of the previous inequalities, we deduce (Corollary 4.2) that if (M^m, g) is minimally immersed in (N^n, h), if $\bar{\rho}^1$ is bounded above and if

$$\min_M (\text{Scal}) > (m-2)\left((m-1)\max_N (\bar{K}^1) + \max_N (\bar{\rho}^1)\right),$$

then (M^m, g) is a sphere of homology. This result can be viewed as a generalization of a theorem of Leung [9] which has shown that if a compact Riemannian manifold (M^m, g) is minimally immersed in a unit sphere and if the scalar curvature satisfies Scal $> m(m-2)$ then it is homeomorphic to an m-dimensional sphere.

2. Preliminaries and notations

Let (M^m, g) be an m-dimensional Riemannian manifold and let ϕ be an isometric immersion of (M^m, g) into an n-dimensional Riemannian manifold (N^n, h) $(n > m)$. The inner product and the norm induced by g and h on the tensors will be denoted, respectively, by $\langle\, ,\, \rangle$ and $|\,|^2$. Moreover, we denote, respectively, by R, ρ, Ric and Scal the curvature tensor, the curvature operator, the Ricci tensor and the scalar curvature of (M^m, g) and by $\bar{R}$, $\bar{K}$, $\bar{\rho}$ and $\overline{\text{Scal}}$ the curvature tensor, the sectional curvature, the curvature operator and the scalar curvature of (N^n, h). We recall that for all vector field $X, Y, Z, W \in \Gamma(TN)$, $\bar{\rho}$ is defined by

$$\langle \bar{\rho}(X \wedge Y), Z \wedge W\rangle = \bar{R}(X, Y, Z, W).$$

Moreover, for all vector field $X, Y \in \Gamma(TM)$, the tensor $\bar{R}_\phi$ will be given by

$$\bar{R}_\phi(X, Y) = \sum_{i \le m} \bar{R}(\mathrm{d}\phi(X), \mathrm{d}\phi(e_i), \mathrm{d}\phi(Y), \mathrm{d}\phi(e_i)),$$

where $(e_i)_{1\le i\le m}$ is an orthonormal frame on M. On the other hand, for all $x \in N$, we denote, respectively, $\bar{K}^1(x)$ and $\bar{K}^0(x)$ the largest sectional curvature and the smallest sectional curvature at x and $\bar{\rho}^1(x)$ and $\bar{\rho}^0(x)$ the largest eigenvalue and the smallest eigenvalue of the curvature operator. Then, it is easy to see that

$$\bar{\rho}^0(x) \le \bar{K}^0(x) \le \bar{K}^1(x) \le \bar{\rho}^1(x). \tag{2}$$

Now, let B be the second fundamental form of the immersion ϕ and let H be the mean curvature vector defined by $H = (1/m)\,\text{trace}\, B$. The Gauss equation tells us that for any vector field $X, Y, Z, W \in \Gamma(TM)$, we have

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + \langle B(X, Z), B(Y, W)\rangle - \langle B(X, W), B(Y, Z)\rangle. \tag{3}$$

For the sake of completeness, we need now to recall briefly some definitions and properties about p-forms. Let $(e_i)_{1\le i\le m}$ be a local orthonormal frame. Throughout this paper, for all q-tensor T, we will write $T_{i_1,\dots,i_q}$ instead of $T(e_{i_1}, \dots, e_{i_q})$ and then the inner product of two p-forms ω and θ of (M^m, g) will be

$$\langle \omega, \theta\rangle = \frac{1}{p!} \sum_{1\le i_1,\dots,i_p\le m} \omega_{i_1,\dots,i_p}\theta_{i_1,\dots,i_p}.$$

The inner product (or contraction) $i(X)\omega$ of a p-form ω with a vector field X on M is a $p-1$-form, defined by

$$(i(X))\omega(X_1,\dots,X_{p-1})=\omega(X,X_1,\dots,X_{p-1})\quad\forall X_1,\dots,X_{p-1}\in\Gamma(TM).$$

More generally, if $X_1,\dots,X_q\in\Gamma(TM)$, then the inner product of the p-form ω with the q-tensor $X_1\wedge\cdots\wedge X_q$ is the $p-q$-form defined by

$$(i(X_1\wedge\cdots\wedge X_q)\omega)(Y_1,\dots,Y_{p-q})$$
$$=\omega(X_q,\dots,X_1,Y_1,\dots,Y_{p-q})\quad\forall Y_1,\dots,Y_{p-q}\in\Gamma(TM).$$

Recall some elementary facts about inner and exterior products. Let ω and θ be, respectively, a p-form and a q-form and let X be a vector field on M, then

$$i(X)(\omega\wedge\theta)=i(X)\omega\wedge\theta+(-1)^p\omega\wedge i(X)\theta$$

and if $X^\star$ is the dual 1-form of the vector field X with respect to g, then $i(X)$ is in fact the adjoint of left exterior multiplication by $X^\star$, that is

$$\langle i(X)(\omega),\theta\rangle=\langle\omega,X^\star\wedge\theta\rangle.$$

If M is orientable, we also need the following relation between the inner product and the Hodge operator $\star$ on (M^m,g) (see, for instance, [4]):

$$i(X)(\star\omega)=(-1)^p\star(X^\star\wedge\omega).$$

On the other hand, if α is a 1-form which is real valued and β is a 1-form which is valued in a vector bundle, we define the 2-tensor $\alpha\vee\beta$ by

$$(\alpha\vee\beta)(X,Y)=\alpha(X)\beta(Y)+\alpha(Y)\beta(X).\tag{4}$$

We denote now by d, $\mathrm{d}^\star$, ∇ and $\nabla^\star$, respectively, the exterior differential and the codifferential acting on p-forms, the covariant derivative of (M^m,g) extended to p-forms and its adjoint with respect to g. The Hodge–de Rham Laplacian Δ acting on p-forms is given by

$$\Delta\omega=\mathrm{d}\,\mathrm{d}^\star\omega+\mathrm{d}^\star\,\mathrm{d}\omega.$$

To compare this Laplacian to the “rough” Laplacian $\nabla^\star\nabla$, one has the Weitzenböck formula, reading as

$$\Delta\omega=\nabla^\star\nabla\omega+\mathcal{R}_p(\omega)\quad\forall\omega\in\Lambda^p(M).$$

Here $\mathcal{R}_p\in\mathrm{End}(\Lambda^p(M))$ is a bundle endomorphism, given by

$$\mathcal{R}_p(\omega)(X_1,\dots,X_p)$$
$$=\sum_{ij}(-1)^i[R(e_j,X_i)\omega](e_j,X_1,\dots,\hat{X}_i,\dots,X_p)\quad\forall X_1,\dots,X_p\in\Gamma(TM),$$

where $(e_i)_{1\le i\le m}$ is a local orthonormal frame and

$$R(X,Y)\omega=\nabla_{[X,Y]}\omega-[\nabla_X,\nabla_Y]\omega\quad\forall X,Y\in\Gamma(TM).$$

An easy consequence of the Weitzenböck formula is the following:

$$\frac{1}{2}\Delta|\omega|^2 = \langle \Delta\omega, \omega\rangle - |\nabla\omega|^2 - \langle \mathcal{R}_p(\omega), \omega\rangle. \tag{5}$$

In the sequel, we need to explicit the expression of $\mathcal{R}_p$. A straightforward calculation gives us

$$\langle \mathcal{R}_p(\omega), \omega\rangle = \sum_{i,j} \mathrm{Ric}_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle - \frac{1}{2}\sum_{i,j,k,l} R_{ijkl}\langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \tag{6}$$

and the last term is zero when $p = 1$.

3. Geometry of submanifolds having a parallel p-form

The first result of this section is the following theorem.

Theorem 3.1. *Let (M^m, g) be an m-dimensional Riemannian manifold admitting a non-trivial parallel p-form $(1 \le p \le m)$ and let (N^n, h) be an n-dimensional Riemannian manifold $(n > m)$. If for any $x \in (N^n, h)$, we have*

$$(m-1)\bar{K}^1(x) < (p-1)\bar{\rho}^0(x) \tag{7}$$

then, there is no minimal immersion from (M^m, g) into (N^n, h).

Remark 3.1.

1. For $p = 2$ and for even dimensional manifold (M^m, g), the pinching condition (7) can be refomulate as the negativity of the isotropic curvature [6]. If (M^m, g) is Kaehlerian, this condition (7) is nothing but that obtained by El Soufi [5] (Theorem 3.2).
2. From relation (2), we see that this theorem is of interest only if the sectional curvature of (N^n, h) is negative. For the hyperbolic space $\mathbb{H}^n$, the condition (7) is always satisfied for $p < m$ and then there is no minimal immersion of a manifold having a parallel p-form $(1 \le p \le m-1)$ into $\mathbb{H}^n$. However, the embeddings of $\mathbb{H}^m$ in $\mathbb{H}^n$ $(m < n)$ are totally geodesic, and taking the volume form of $\mathbb{H}^m$, we see that (7) is not satisfied for $p = m$.

In the following theorem, we obtain the same conclusion as in Theorem 3.1 with a new pinching condition where (N^n, h) is not necessarily of negative sectional curvature (in fact, there is no condition on $\bar{K}^1$).

Theorem 3.2. *Let (M^m, g) be an m-dimensional Riemannian manifold admitting a non-trivial parallel p-form $(1 \le p \le m)$ and let (N^n, h) be an n-dimensional Riemannian manifold $(n > m)$. If for any $x \in (N^n, h)$, we have*

$$\overline{\mathrm{Scal}}(x) < (n-m)(n+m-1)\bar{K}^0(x) + (p(p-1) + (m-p)(m-p-1))\bar{\rho}^0(x) \tag{8}$$

then, there is no minimal immersion from (M^m, g) into (N^n, h).

Remark 3.2. We will see in the proof that this theorem is of interest only if the smallest sectional curvature is negative that is $\bar{K}^0(x) < 0$ for all $x \in N$. For instance, let us consider the space $N^n = \mathbb{H}^r \times \mathbb{S}^s$ where $n = r+s$. Then $\overline{\mathrm{Scal}} = -r(r-1)+s(s-1)$, $\bar{\rho}^1 = \bar{K}^1 = 1$ and $\bar{\rho}^0 = \bar{K}^0 = -1$. Now, let (M^m, g) be a Riemannian manifold of even dimension m and let $p = m/2$. Then we have $\overline{\mathrm{Scal}} - (n-m)(n+m-1)\bar{K}^0 - (p(p-1)+(m-p)(m-p-1))\bar{\rho}^0 = 2(s^2 + rs - s - m^2/4)$ and it is easy to see that if r and m are great enough (for instance, for a fixed s put $m = r$ great enough) then the condition (8) is satisfied and the conclusion of Theorem 3.2 holds for this example.

Proof of Theorem 3.1. Let ϕ be a minimal immersion of (M^m, g) into (N^n, h) and assume that M has a nontrivial parallel p-form ω. Then $\langle \mathcal{R}_p(\omega), \omega\rangle = 0$ and from (6) we deduce

$$0 = \sum_{i,j} \mathrm{Ric}_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle - \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \tag{9}$$

and the last term is zero for $p = 1$. Now from the Gauss formula (3), we obtain

$$\sum_{i,j} \mathrm{Ric}_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle = \sum_{i,j} (\bar{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle + m \sum_{i,j} \langle B_{ij}, H\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle - \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle \tag{10}$$

and

$$\begin{aligned}\sum_{i,j,k,l} & R_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \\ &= \sum_{i,j,k,l} \bar{R}_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle + \sum_{i,j,k,l} \langle B_{ik}, B_{jl}\rangle \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \\ &\quad - \sum_{i,j,k,l} \langle B_{il}, B_{jk}\rangle \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \\ &= \sum_{i,j,k,l} \bar{R}_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle + 2 \sum_{i,j,k,l} \langle B_{ik}, B_{jl}\rangle \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle\end{aligned} \tag{11}$$

by reporting (10) and (11) in (9), we get

$$\begin{aligned} 0 = &\sum_{i,j} (\bar{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle + m \sum_{i,j} \langle B_{ij}, H\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle \\ &- \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle - \frac{1}{2} \sum_{i,j,k,l} \bar{R}_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \\ &- \sum_{i,j,k,l} \langle B_{ik}, B_{jl}\rangle \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \end{aligned} \tag{12}$$

and the two last terms are zero for $p = 1$. Now, put $\mathcal{B}^{+}(\omega) = \sum_{i\leq m} i(e_i^{\star})\omega \wedge B(e_i, \cdot)$. The computation of the square of the norm of $\mathcal{B}^{+}(\omega)$ gives

$$|\mathcal{B}^{+}(\omega)|^2 = \frac{1}{p!} \sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j}} \langle (i(e_i^{\star})\omega \wedge B(e_i,\cdot))_{i_1,\dots,i_p}, (i(e_j^{\star})\omega \wedge B(e_j,\cdot))_{i_1,\dots,i_p}\rangle$$
$$= \frac{1}{p!} \sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s,t}} (-1)^{s+t} \langle B_{ii_s}, B_{ji_t}\rangle \omega_{ii_1,\dots,\hat{i}_s,\dots,i_p}\omega_{ji_1,\dots,\hat{i}_t,\dots,i_p},$$

where the indices with ^ are omitted. Then

$$|\mathcal{B}^{+}(\omega)|^2 = \frac{1}{p!} \sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s}} \langle B_{ii_s}, B_{ji_s}\rangle \omega_{ii_1,\dots,\hat{i}_s,\dots,i_p}\omega_{ji_1,\dots,\hat{i}_s,\dots,i_p}$$
$$+\frac{1}{p!} \sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s\neq t}} (-1)^{s+t} \langle B_{ii_s}, B_{ji_t}\rangle \omega_{ii_1,\dots,\hat{i}_s,\dots,i_p}\omega_{ji_1,\dots,\hat{i}_t,\dots,i_p}$$
$$= \frac{1}{(p-1)!} \sum_{\substack{1\leq i_1,\dots,i_{p-1}\leq m\\ i,j,k}} \langle B_{ik}, B_{jk}\rangle \omega_{ii_1,\dots,i_{p-1}}\omega_{ji_1,\dots,i_{p-1}}$$
$$-\frac{1}{p!} \sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s<t}} \langle B_{ii_s}, B_{ji_t}\rangle \omega_{ii_ti_1,\dots,\hat{i}_s,\dots,\hat{i}_t,\dots,i_p}\omega_{ji_si_1,\dots,\hat{i}_s,\dots,\hat{i}_t,\dots,i_p}$$
$$-\frac{1}{p!} \sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s>t}} \langle B_{ii_s}, B_{ji_t}\rangle \omega_{ii_ti_1,\dots,\hat{i}_t,\dots,\hat{i}_s,\dots,i_p}\omega_{ji_si_1,\dots,\hat{i}_t,\dots,\hat{i}_s,\dots,i_p}$$
$$= \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle$$
$$-\frac{1}{(p-2)!} \sum_{\substack{1\leq i_1,\dots,i_{p-2}\leq m\\ i,j,k,l}} \langle B_{ik}, B_{jl}\rangle \omega_{ili_1,\dots,i_{p-2}}\omega_{jki_1,\dots,i_{p-2}}$$
$$= \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle$$
$$+\frac{1}{(p-2)!} \sum_{\substack{1\leq i_1,\dots,i_{p-2}\leq m\\ i,j,k,l}} \langle B_{ik}, B_{jl}\rangle \omega_{iji_1,\dots,i_{p-2}}\omega_{kli_1,\dots,i_{p-2}}$$
$$= \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle + \sum_{i,j,k,l} \langle B_{ik}, B_{jl}\rangle \langle i(e_j\wedge e_i)\omega, i(e_l\wedge e_k)\omega\rangle.$$

Note that if M is Kaehlerian and ω is the Kaehler form then $|\mathcal{B}^+(\omega)|^2 = |B^+|^2$, where B^+ is the holomorphic part of B (i.e. $B^+(X, Y) = (1/2)(B(X, Y) + B(JX, JY))$ where $\omega(X, Y) = \langle JX, Y\rangle$).

Now, combining the above relation with (12), we obtain

$$|\mathcal{B}^+(\omega)|^2 = m\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle + \sum_{i,j}(\bar{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle - \frac{1}{2}\sum_{i,j,k,l}\bar{R}_{ijkl}\langle i(e_j\wedge e_i)\omega, i(e_l\wedge e_k)\omega\rangle, \tag{13}$$

where the last term is zero if $p = 1$. Putting $X^{i_1,\dots,i_p} = \sum_{i\le m}\omega_{ii_1,\dots,i_{p-1}}e_i$ and $\theta^{i_1,\dots,i_{p-2}} = (1/2)\sum_{1\le i,j\le m}\omega_{iji_1,\dots,i_{p-2}}e_i^\star\wedge e_j^\star$, we have

$$\begin{aligned}\sum_{i,j}(\bar{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle &= \frac{1}{(p-1)!}\sum_{i_1,\dots,i_{p-1}}\bar{R}_\phi(X^{i_1,\dots,i_p}, X^{i_1,\dots,i_p})\\ &\le \frac{m-1}{(p-1)!}\bar{K}^1(\phi(x))\sum_{i_1,\dots,i_{p-1}}|X^{i_1,\dots,i_p}|^2\\ &= p(m-1)\bar{K}^1(\phi(x))|\omega|^2\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\frac{1}{2}\sum_{i,j,k,l}\bar{R}_{ijkl}\langle i(e_j\wedge e_i)\omega, i(e_l\wedge e_k)\omega\rangle\\ &\quad= \frac{2}{(p-2)!}\sum_{i_1,\dots,i_{p-2}}\bar{\rho}(\theta^{i_1,\dots,i_{p-2}}, \theta^{i_1,\dots,i_{p-2}})\\ &\quad\ge \frac{2}{(p-2)!}\bar{\rho}^0(\phi(x))\sum_{i_1,\dots,i_{p-2}}|\theta^{i_1,\dots,i_{p-2}}|^2\\ &\quad= \frac{1}{(p-2)!}\bar{\rho}^0(\phi(x))\sum_{i_1,\dots,i_p}\omega^2_{i_1,\dots,i_p} = p(p-1)\bar{\rho}^0(\phi(x))|\omega|^2,\end{aligned} \tag{15}$$

consequently, for all $p\in\{1,\dots,m\}$, it follows from (13) and the fact that $H = 0$:

$$|\mathcal{B}^+(\omega)|^2(x) \le p(m-1)\bar{K}^1(\phi(x))|\omega|^2 - p(p-1)\bar{\rho}^0(\phi(x))|\omega|^2.$$

From this we conclude that if $(p-1)\bar{\rho}^0(x) > (m-1)\bar{K}^1(x)$ holds for any $x\in N$, then there is no minimal immersion from (M^m, g) into (N^n, h). □

Remark 3.3. In Theorem 3.1, the nonpositivity of the curvature operator of (N^n, h) is not required. In [2] Corlette proved (Theorem 4.1) a similar result for harmonic maps ϕ from (M^m, g) to (N^n, h) (in fact he assumes that ϕ is a twisted harmonic map which is not necessary) by assuming only the nonpositivity of the curvature operator of (N^n, h) without pinching condition. Indeed, under this hypothesis he proved that if (M^m, g) is compact and has a parallel p-form then $\nabla^\star(\mathrm{d}\phi \wedge \omega) = 0$ (here ∇ is the pullback of the Levi–Civita connection of TN). Note that Corlette does not obtain a result of nonexistence. Moreover if ϕ is a minimal isometric immersion, then ϕ is a harmonic map and the theorem of Corlette can be proved without the compacity of M and a short computation shows that $\nabla^\star(\mathrm{d}\phi \wedge \omega) = -\mathcal{B}^+(\omega)$.

Proof of Theorem 3.2. From relation (13) and the inequality (15) of the previous proof, we deduce that if ϕ is a minimal immersion, we have

$$|\mathcal{B}^+(\omega)|^2 \leq \sum_{i,j} (\bar{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle - p(p-1)\bar{\rho}^0(\phi(x))|\omega|^2.$$

Since M is locally oriented, we can define locally the Hodge operator $\star$ and the $m-p$-form $\star\omega$. But $|\mathcal{B}^+(\star\omega)|^2$ is independent of the choice of the orientability and is consequently globally defined. Then we have

$$|\mathcal{B}^+(\omega)|^2 + |\mathcal{B}^+(\star\omega)|^2 \leq \sum_{i,j} (\bar{R}_\phi)_{ij} (\langle i(e_i)\omega, i(e_j)\omega\rangle + \langle i(e_i)\star\omega, i(e_j)\star\omega\rangle) \\ -(p(p-1) + (m-p)(m-p-1))\bar{\rho}^0(\phi(x))|\omega|^2. \qquad (16)$$

Now using the properties about inner and exterior products recalled in the first section, we have

$$\begin{aligned} &\sum_{i,j} ((\bar{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle + (\bar{R}_\phi)_{ij} \langle i(e_i)\star\omega, i(e_j)\star\omega\rangle) \\ &\quad = \sum_{i,j} ((\bar{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle + (\bar{R}_\phi)_{ij} \langle e_i^\star \wedge \omega, e_j^\star \wedge \omega\rangle) \\ &\quad = \sum_{i,j} ((\bar{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle + (\bar{R}_\phi)_{ij} \langle i(e_j)(e_i^\star \wedge \omega), \omega\rangle) \\ &\quad = \sum_{i,j} (\bar{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle + \sum_i (\bar{R}_\phi)_{ii}|\omega|^2 - \sum_{i,j} (\bar{R}_\phi)_{ij} \langle e_i^\star \wedge i(e_j)\omega, \omega\rangle \\ &\quad = \sum_i (\bar{R}_\phi)_{ii}|\omega|^2. \end{aligned} \qquad (17)$$

A straightforward computation gives

$$\sum_i (\bar{R}_\phi)_{ii} \leq \overline{\mathrm{Scal}}(\phi(x)) - (n-m)(n+m-1)\bar{K}^0(\phi(x))$$

and from (16) and (17) we deduce that for all $x \in M$:

$$0 \leq \overline{\mathrm{Scal}}(\phi(x)) - ((n-m)(n+m-1)\bar{K}^0 + (p(p-1) \\ + (m-p)(m-p-1))\bar{\rho}^0)(\phi(x)).$$

Consequently if

$$\overline{\mathrm{Scal}}(x) < (n-m)(n+m-1)\bar{K}^0(x) + (p(p-1) + (m-p)(m-p-1))\,\bar{\rho}^0(x)$$

for all $x \in N$, there is no minimal immersion from (M^m, g) into (N^n, h). Using the inequalities (2), we can easily see that the above condition implies

$$\left(\frac{m(m-1)}{p(p-1)+(m-p)(m-p-1)}\right)\bar{K}^0(x) < \bar{\rho}^0(x) \leq \bar{K}^0(x)$$

for all $x \in N$. And Theorem 3.2 is of interest only for $\bar{K}^0 < 0$. □

To finish this section, we study the particular case where the ambient space is the complex hyperbolic space $\mathbb{CH}^n(c)$ with constant holomorphic curvature equal to c ($c < 0$). Let us recall that in this case the curvature tensor of $\mathbb{CH}^n(c)$ has the expression:

$$\bar{R}(X, Y, Z, W) = \tfrac{1}{4}c(\langle X \wedge Y, Z \wedge W\rangle + \langle X \wedge Y, JZ \wedge JW\rangle + 2\langle X, JY\rangle\langle Z, JW\rangle) \quad (18)$$

for all $X, Y, Z, W \in \Gamma(T\mathbb{CH}^n(c))$. Here J denotes the complex structure of $\mathbb{CH}^n(c)$. For any isometric immersion ϕ of a Riemannian manifold (M^m, g) into $\mathbb{CH}^n(c)$, we define the (1, 1)-tensor J_ϕ on M by $J_\phi X = \sum_{i\leq m}\langle J\,\mathrm{d}\phi(X), \mathrm{d}\phi(e_i)\rangle e_i$ $\forall X \in \Gamma(TM)$ and for all orthonormal frame $(e_i)_{1\leq i\leq m}$. Recall that the immersion ϕ is said to be totally real if $J_\phi \equiv 0$.

For this kind of immersions we have the following theorem.

Theorem 3.3. *Let (M^m, g) be an m-dimensional Riemannian manifold admitting a nontrivial parallel p-form ($p \geq 1$). Then there is no minimal totally real immersion of (M^m, g) into $\mathbb{CH}^n(c)$.*

Proof. Let ϕ be a minimal immersion of (M^m, g) into $\mathbb{CH}^n$ and assume that M has a nontrivial parallel p-form ω. Then from (13) we have

$$|\mathcal{B}^+(\omega)|^2 = \sum_{i,j}(\bar{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle - \frac{1}{2}\sum_{i,j,k,l}\bar{R}_{ijkl}\langle i(e_j\wedge e_i)\omega, i(e_l\wedge e_k)\omega\rangle.$$

Now the conclusion follows from a straightforward computation. Using (18) the above equality becomes

$$\begin{aligned}|\mathcal{B}^+(\omega)|^2 = \frac{c}{4}\Bigg(&p(m-p)|\omega|^2 + 3\sum_{i,j\leq m}\langle J_\phi e_i, J_\phi e_j\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle \\ &- \sum_{i,j\leq m}\langle i(e_j\wedge e_i)\omega, i(J_\phi e_j \wedge J_\phi e_i)\omega\rangle \\ &- \sum_{i,j\leq m}\langle i(e_i\wedge J_\phi e_i)\omega, i(e_j\wedge J_\phi e_j)\omega\rangle\Bigg),\end{aligned}$$

where $(e_i)_{1\leq i\leq m}$ is a local orthonormal frame. □

4. Geometry of submanifolds with $b_p(M) \neq 0$

Let (M^m, g) and (N^n, h) be two Riemannian manifolds and assume that (M^m, g) is compact. We use the same notations as in the previous sections. Moreover in this section, $k(x)$ denotes the smallest eigenvalue at x of the Ricci curvature of (M^m, g) and we put $k_0 = \min_M (k(x))$. On the other hand, if $\bar{K}^1$ is bounded above, we will set $\bar{K}^1_{\max} = \max_N (\bar{K}^1)$ and $\bar{\rho}^1_{\max} = \max_N (\bar{\rho}^1)$. The first result is the following theorem.

Theorem 4.1. *Let (M^m, g) be a compact Riemannian manifold of dimension $m \geq 2$ so that $b_p(M) \neq 0$ for some $p \geq 1$. Then for any isometric immersion ϕ from (M^m, g) into an n-dimensional Riemannian manifold (N^n, h), there exists at least a point $x \in M$ so that*

$$\frac{m}{\sqrt{p}}\left(\frac{p-1}{p}\right)|B(x)||H(x)| \geq k(x) - \left(\frac{p-1}{p}\right)((m-1)\bar{K}^1 + \bar{\rho}^1)(\phi(x)). \tag{19}$$

The following corollary is an immediate consequence of this theorem.

Corollary 4.1. *Let (M^m, g) be a compact Riemannian manifold of dimension $m \geq 2$ minimally immersed in an n-dimensional Riemannian manifold (N^n, h) $(n > m)$. If $\bar{\rho}^1$ is bounded above and if for an integer p so that $1 \leq p \leq m/2$, we have*

$$k_0 > \left(\frac{p-1}{p}\right)((m-1)\bar{K}^1_{\max} + \bar{\rho}^1_{\max}),$$

then $b_q(M) = 0$ for $q \in \{1, \ldots, p\}$.

Remark 4.1. The inequality (19) is an equality at each point for the standard embedding of $\mathbb{S}^p(\sqrt{p/m}) \times \mathbb{S}^{k_1}(\sqrt{k_1/m}) \times \cdots \times \mathbb{S}^{k_q}(\sqrt{k_q/m})$ into $\mathbb{S}^{m+r}$ where $p + k_1 + \cdots + k_q = m$ with $k_i \geq p$ $(1 \leq i \leq q)$. For the case $p = 2$, (19) is also an equality at each point for the standard embedding from $\mathbb{C}P^q$ with holomorphic curvature $2q/(q+1)$ into $\mathbb{S}^{q^2+2q}$.

Theorem 4.1 is a generalization of a result obtained by El Soufi (see Theorem 3.1 of [5]) in the particular case $p = 2$.

To prove Theorem 4.1 we need the following proposition which gives an estimate of the term $\langle \mathcal{R}_p(\omega), \omega \rangle$ for any p-form ω.

Proposition 4.1. *Let (M^m, g) be an m-dimensional compact Riemannian manifold isometrically immersed in an n-dimensional Riemannian manifold (N^n, h). Then for any $p \in \{1, \ldots, m\}$ and for any p-form ω of M, we have for all $x \in M$*

$$\langle \mathcal{R}_p(\omega), \omega \rangle \geq p\left(pk(x) - (p-1)((m-1)\bar{K}^1 + \bar{\rho}^1)(\phi(x)) - m\left(\frac{p-1}{\sqrt{p}}\right)|H(x)||B(x)|\right)|\omega|^2.$$

Before proving this proposition, we introduce the following p-tensor associated to any p-form ω of M and the isometric immersion ϕ

$$\mathcal{B}^-(\omega)=\frac{1}{(p-2)!}\sum_{i,i_1,\dots,i_{p-2}\le m}\Big(\big(i(e_i\wedge e_{i_1}\wedge\cdots\wedge e_{i_{p-2}})\omega\big)\vee B(e_i,\cdot)\Big)\otimes(e^\star_{i_1}\wedge\cdots\wedge e^\star_{i_{p-2}}),$$

where $(e_i)_{1\le i\le m}$ is an orthonormal frame at a point $x\in M$ and $\vee$ denotes the symmetric product defined in the preliminaries (see (4)). It will be convenient to choose the norm $|\mathcal{B}^-(\omega)|$ so that

$$|\mathcal{B}^-(\omega)|^2=\frac{1}{(p-2)!}\sum_{jki_1,\dots,i_{p-2}\le m}|\mathcal{B}^-(\omega)_{jki_1,\dots,i_{p-2}}|^2.$$

Such a tensor has been introduced for the first time in [7] where the second fundamental form is replaced by the Hessian of a function. First note that if ω is a volume form at a point p of M where we have define the Hodge operator $\star$ so that $\omega=\star 1$, we deduce from (20) shown in the proof of Proposition 4.1, that

$$\begin{aligned}\frac{1}{2n}|\mathcal{B}^-(v_g)|^2&=\frac{n-1}{n}\sum_{ijk\le n}\langle B_{ik},B_{jk}\rangle\langle i(e_i)\star 1,i(e_j)\star 1\rangle\\&\quad-\frac{1}{n}\sum_{ijkl\le n}\langle B_{ij},B_{kl}\rangle\langle i(e_l\wedge e_i)\star 1,i(e_k\wedge e_j)\star 1\rangle\\&=\frac{n-1}{n}\sum_{ijk\le n}\langle B_{ik},B_{jk}\rangle\langle e^\star_i,e^\star_j\rangle-\frac{1}{n}\sum_{ijkl\le n}\langle B_{ij},B_{kl}\rangle\langle e^\star_l\wedge e^\star_i,e^\star_k\wedge e^\star_j\rangle\\&=|B|^2-n|H|^2=|B-H\otimes g|^2.\end{aligned}$$

In other words we obtain the square of the umbilicity tensor. Moreover if α denotes the Kaehler form of a Kaehlerian manifold, a straightforward calculation gives

$$|\mathcal{B}^-(\alpha)|^2=4|B^-|^2,$$

where B^- is the anti-holomorphic part of B (i.e. $B^-(X,Y)=(1/2)(B(X,Y)-B(JX,JY))$ where $\alpha(X,Y)=\langle JX,Y\rangle$).

On the other hand we will see in the proof of Theorem 4.1 that $\mathcal{B}^-(\omega)$ is vanishing identically for the standard embedding of $\mathbb{S}^p(\sqrt{p/m})\times\mathbb{S}^{k_1}(\sqrt{k_1/m})\times\cdots\times\mathbb{S}^{k_q}(\sqrt{k_q/m})$ into $\mathbb{S}^{m+r}$ where $p+k_1+\cdots+k_q=m$ with $k_i\ge p$ $(1\le i\le q)$.

Proof of Proposition 4.1. For $p=1$ and from relation (6), the equality of the proposition is obvious. Suppose now that $p\ge 2$. Let $x\in M$ and let $(e_i)_{1\le i\le m}$ be a local orthonormal frame in a neighborhood of x. We have

$$\begin{aligned}\frac{(p-2)!}{2}|\mathcal{B}^-(\omega)|^2&=\frac{1}{2}\sum_{i,j,i_1,\dots,i_{p-2}}|\mathcal{B}^-(\omega)_{i,j,i_1,\dots,i_{p-2}}|^2\\&=\frac{1}{2}\sum_{i,j,k,l,i_1,\dots,i_{p-2}}\langle(i(e_i\wedge e_{i_1}\wedge\cdots\wedge e_{i_{p-2}})\omega\vee B(e_i,\cdot))_{kl},\\&\quad(i(e_j\wedge e_{i_1}\wedge\cdots\wedge e_{i_{p-2}})\omega\vee B(e_j,\cdot))_{kl}\rangle\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,k,l,i_1,\dots,i_{p-2}} (i(e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_{p-2}})\omega)_k \\
&\quad (i(e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_{p-2}})\omega)_k \langle B_{il}, B_{jl}\rangle \\
&\quad + \sum_{i,j,k,l,i_1,\dots,i_{p-2}} (i(e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_{p-2}})\omega)_k \\
&\quad (i(e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_{p-2}})\omega)_l \langle B_{il}, B_{jk}\rangle \\
&= \sum_{i,j,k,l,i_1,\dots,i_{p-2}} \langle B_{il}, B_{jl}\rangle \omega_{iki_1,\dots,i_{p-2}} \omega_{jki_1,\dots,i_{p-2}} \\
&\quad + \sum_{i,j,k,l,i_1,\dots,i_{p-2}} \langle B_{il}, B_{jk}\rangle \omega_{iki_1,\dots,i_{p-2}} \omega_{jli_1,\dots,i_{p-2}} \\
&= (p-1)! \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle \\
&\quad -(p-2)! \sum_{i,j,k,l} \langle B_{ij}, B_{kl}\rangle \langle i(e_l \wedge e_i)\omega, i(e_k \wedge e_j)\omega\rangle.
\end{aligned}
$$

Finally, we have proved that

$$
\begin{aligned}
\frac{1}{2}|\mathcal{B}^-(\omega)|^2 &= (p-1) \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle \\
&\quad - \sum_{i,j,k,l} \langle B_{ik}, B_{jl}\rangle \langle i(e_i \wedge e_j)\omega, i(e_k \wedge e_l)\omega\rangle.
\end{aligned}
\tag{20}
$$

Now, combining this with relations (10) and (11) and using the expression of $\langle \mathcal{R}_p(\omega), \omega\rangle$ (see (6)), we get

$$
\begin{aligned}
\frac{1}{2}|\mathcal{B}^-(\omega)|^2 &= \langle \mathcal{R}_p(\omega), \omega\rangle - p \sum_{i,j} \mathrm{Ric}_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle \\
&\quad +(p-1) \sum_{i,j} (\bar{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle \\
&\quad +\frac{1}{2} \sum_{i,j,k,l} \bar{R}_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \\
&\quad +m(p-1) \sum_{i,j} \langle B_{ij}, H\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle.
\end{aligned}
\tag{21}
$$

From the hypotheses on the curvature of N and by techniques already used in the proof of Theorem 3.1 (see (14) and (15)) we deduce that

$$\frac{1}{2}|\mathcal{B}^-(\omega)|^2 \leq \langle \mathcal{R}_p(\omega), \omega\rangle - p\sum_{i,j} \mathrm{Ric}_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + p(p-1)\left((m-1)\bar{K}^1(x) + \bar{\rho}^1(x)\right)|\omega|^2 + m(p-1)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle. \quad (22)$$

Now, let us estimate the last term. For this, assume that at the point $x \in M$, $(e_i)_{1\leq i\leq m}$ diagonalizes the symmetric tensor $\langle B(X, Y), H\rangle$. We have

$$\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle = \frac{1}{(p-1)!}\sum_{i,i_1,\dots,i_{p-1}}\langle B_{ii}, H\rangle\omega^2_{ii_1,\dots,i_{p-1}}$$
$$= \frac{1}{p!}\sum_{i,i_1,\dots,i_{p-1}}(\langle B_{ii}, H\rangle + \langle B_{i_1i_1}, H\rangle + \cdots + \langle B_{i_{p-1}i_{p-1}}, H\rangle)\omega^2_{ii_1,\dots,i_{p-1}}$$
$$\leq \frac{1}{p!}\sum_{i,i_1,\dots,i_{p-1}}(|B_{ii}| + |B_{i_1i_1}| + \cdots + |B_{i_{p-1}i_{p-1}}|)|H|\omega^2_{ii_1,\dots,i_{p-1}}$$
$$\leq \frac{\sqrt{p}}{p!}\sum_{i,i_1,\dots,i_{p-1}}|B||H|\omega^2_{ii_1,\dots,i_{p-1}}.$$

Finally we have proved

$$\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle \leq \sqrt{p}|B||H||\omega|^2. \quad (23)$$

Since $\mathrm{Ric} \geq kg$, we have $\sum_{i,j}\mathrm{Ric}_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle \geq pk|\omega|^2$, and we deduce from this, (22) and (23), the inequality of Proposition 4.1. □

The proof of Theorem 4.1 is now an immediate consequence of Proposition 4.1.

Proof of Theorem 4.1. Since $b_p(M) \neq 0$, there exists a nontrivial p-form ω so that $\Delta\omega = 0$. And from the Weitzenböck formula (5), we deduce

$$\int_M \langle \mathcal{R}_p(\omega), \omega\rangle \leq 0. \quad (24)$$

Now, applying the estimate of Proposition 4.1 we get the desired inequality. □

We can show a similar result to Theorem 4.1, with the scalar curvature Scal of (M^m, g) instead of the Ricci curvature.

Theorem 4.2. *Let (M^m, g) be a compact Riemannian manifold of dimension $m \geq 2$ so that $b_p(M) \neq 0$ for a $p \geq 1$. Then for any isometric immersion ϕ from (M^m, g) into an n-dimensional Riemannian manifold (N^n, h), there exists at least a point $x \in M$ so that*

$$m\left(\frac{p-1}{\sqrt{p}}+\frac{m-p-1}{\sqrt{m-p}}\right)|B(x)||H(x)|$$
$$\geq \mathrm{Scal}(x)-(m-2)((m-1)\bar{K}^1+\bar{\rho}^1)(\phi(x)). \quad (25)$$

We immediately deduce the following corollary.

Corollary 4.2. *Let (M^m, g) be a compact Riemannian manifold of dimension $m \geq 2$ minimally immersed into an n-dimensional Riemannian manifold (N^n, h) $(n > m)$. If $\bar{\rho}^1$ is bounded above and if*

$$\min_M(\mathrm{Scal}) > (m-2)\left((m-1)\bar{K}^1_{\max}+\bar{\rho}^1_{\max}\right),$$

then for any $p \in \{1, \ldots, m\}$, we have $b_p(M) = 0$.

Remark 4.2. If $p = m/2$, we can improve (25) to obtain

$$m(m-2)|H(x)|^2 \geq \mathrm{Scal}(x)-(m-2)((m-1)\bar{K}^1+\bar{\rho}^1)(\phi(x)). \quad (26)$$

Theorem 4.2 and the inequality (26) was obtained by El Soufi (Theorems 3.1 and 3.2 of [5]) in the particular case where $m = 4$ and $p = 2$.

On the other hand, note that for $p = m/2$, Theorem 4.1 is not a consequence of Theorem 4.2.

For the same reasons as in the proof of Theorem 3.2, we can choose locally an orientation on M and define locally the Hodge operator $\star$. But for all p-form ω of M, the quantity $\langle \mathcal{R}_{m-p}(\star\omega), \star\omega\rangle$ is globally defined.

Theorem 4.2 is a consequence of the following proposition.

Proposition 4.2. *Let (M^m, g) be an m-dimensional compact Riemannian manifold isometrically immersed in an n-dimensional Riemannian manifold (N^n, h). Then for all $p \in \{1, \ldots, m-1\}$ and for all p-form ω on (M^m, g), we have for all $x \in M$:*

$$\langle \mathcal{R}_p(\omega), \omega\rangle + \frac{p}{m-p}\langle \mathcal{R}_{m-p}(\star\omega), \star\omega\rangle$$
$$\geq p\Bigg(\mathrm{Scal}(x)-(m-2)((m-1)\bar{K}^1+\bar{\rho}^1)(\phi(x))$$
$$-m\left(\frac{p-1}{\sqrt{p}}+\frac{m-p-1}{\sqrt{m-p}}\right)|H(x)||B(x)|\Bigg)|\omega|^2.$$

Remark 4.3. If $p = m/2$, we can improve this inequality (see the proof of Proposition 4.2) to obtain

$$\langle \mathcal{R}_p(\omega), \omega\rangle + \frac{p}{m-p}\langle \mathcal{R}_{m-p}(\star\omega), \star\omega\rangle$$
$$\geq p(\mathrm{Scal}(x)-(m-2)((m-1)\bar{K}^1+\bar{\rho}^1)(\phi(x))-m(m-2)|H(x)|^2)|\omega|^2. \quad (27)$$

And if for $p \geq 1$, $b_p(M) \neq 0$, we get (26).

Proof of Proposition 4.2. From the inequality (22) we obtain that for all $p \geq 1$:

$$\langle \mathcal{R}_p(\omega), \omega\rangle \geq p\sum_{i,j}\mathrm{Ric}_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle - p(p-1)((m-1)\bar{K}^1 + \bar{\rho}^1)(\phi(x))|\omega|^2$$
$$-m(p-1)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle. \quad (28)$$

Since $\star\omega$ is a $(m-p)$-form we have also

$$\langle \mathcal{R}_{m-p}(\star\omega), \star\omega\rangle \geq (m-p)\sum_{i,j}\mathrm{Ric}_{ij}\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle$$
$$-(m-p)(m-p-1)((m-1)\bar{K}^1 + \bar{\rho}^1)(\phi(x))|\omega|^2$$
$$-m(m-p-1)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle. \quad (29)$$

Multiplying (29) by $p/(m-p)$, and summing the obtained inequality with (28), we find

$$\langle \mathcal{R}_p(\omega), \omega\rangle + \frac{p}{m-p}\langle \mathcal{R}_{m-p}(\star\omega), \star\omega\rangle$$
$$\geq p\sum_{i,j}(\mathrm{Ric}_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + \mathrm{Ric}_{ij}\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle$$
$$- p(m-2)((m-1)\bar{K}^1 + \bar{\rho}^1)(\phi(x))|\omega|^2 - m(p-1)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle$$
$$- mp\left(\frac{m-p-1}{m-p}\right)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle.$$

By computations which are similar to (17) we get

$$\sum_{i,j}(\mathrm{Ric}_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + \mathrm{Ric}_{ij}\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle) = \mathrm{Scal}|\omega|^2. \quad (30)$$

Thus

$$\langle \mathcal{R}_p(\omega), \omega\rangle + \frac{p}{m-p}\langle \mathcal{R}_{m-p}(\star\omega), \star\omega\rangle$$
$$\geq p\,\mathrm{Scal}|\omega|^2 - p(m-2)((m-1)\bar{K}^1 + \bar{\rho}^1)(\phi(x))|\omega|^2$$
$$-m(p-1)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle$$
$$-mp\left(\frac{m-p-1}{m-p}\right)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle. \quad (31)$$

(Note that if $p = m/2$, then $p-1 = p((m-p-1)/(m-p))$, and we show with the same arguments as in the proof of (30) that

$$m(p-1)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle$$
$$+mp\left(\frac{m-p-1}{m-p}\right)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle = p(m-2)m|H|^2|\omega|^2$$

and reporting this in (31), we obtain the inequality (27) of the Remark 4.3.)

From the estimate (23), we deduce

$$m(p-1)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle \le m(p-1)\sqrt{p}|H||B||\omega|^2$$

and

$$mp\left(\frac{m-p-1}{m-p}\right)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle \le \frac{mp(m-p-1)}{\sqrt{m-p}}|H||B||\omega|^2,$$

now reporting these inequalities in (31), we find the inequality of the proposition. □

The proof of the theorem is now immediate.

Proof of Theorem 4.2. Let ω be a harmonic p-form. Since the Hodge operator commutes with the Laplacian, then $\star\omega$ is a harmonic $(m-p)$-form, and we have

$$\int_M\left(\langle\mathcal{R}_p(\omega), \omega\rangle + \frac{p}{m-p}\langle\mathcal{R}_{m-p}(\star\omega), \star\omega\rangle\right) \le 0$$

and the theorem follows from Proposition 4.2. □

Remark 4.4. We can improve all the results of this section by considering the particular case of submanifolds of the complex projective space $\mathbb{C}P^n(c)$ $(c > 0)$. We just need to compute in (21) the terms with the curvature tensor of the complex projective space. Then we obtain the same statements as previously by replacing $(m-1)\bar{K}^1+\bar{\rho}^1$ by $c/8((m-1)+3\|J_\phi\|^2)$ where for all $x \in M$, $\|J_\phi\|(x) = \sup\{|J_\phi(X)|/X \in T_xM \text{ and } |X| = 1\}$. In particular, if (M^m, g) is an m-dimensional compact Riemannian manifold minimally immersed in $\mathbb{C}P^n(c)$ and if

$$\min_M(\text{Scal}) > \tfrac{1}{8}(c(m-2)((m-1)+3\max_M(\|J_\phi\|^2))),$$

then for any $p \in \{1, \dots, m\}$, we have $b_p(M) = 0$.

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